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# Infinitesimal Deformations of Cusp Singularities

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Introduction. The purpose of this article is to compute infinitesimal deformations  $\pi^1$  of cusp singularities of two dimension. Let  $T$  be a cusp singularity,  $C$  the exceptional set of the minimal resolution of  $T$ ,  $r$  the number of irreducible components of  $C$ . Then  $C$  is a (reduced) cycle of  $r$  rational curves. Our main consequence is that  $\dim \pi^1$  is equal to  $r - C^2$  if  $C^2 \leq -5$ . This has been conjectured by Behnke [1]. After completing this work, I was informed that Behnke [2] solved this in a manner slightly different from ours.

## §1 Definitions

(1.1) Let  $M$  be a complete module in a real quadratic field  $K$ ,  $U^+(M)$  the group of all totally positive units keeping  $M$  invariant by multiplication,  $V$  an infinite cyclic subgroup of  $U^+(M)$ . We define a subgroup  $G(M, V)$  of  $SL(2, \mathbb{R})$  by

$$G(M, V) = \left\{ \begin{pmatrix} v & m \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{R}); v \in V, m \in M \right\}.$$

We define an action of  $G(M, V)$  on the product  $\mathbb{H} \times \mathbb{H}$  of two upper half planes by

$$\begin{pmatrix} v & m \\ 0 & 1 \end{pmatrix} : (z_1, z_2) \rightarrow (vz_1 + m, v'z_2 + m')$$

where  $v'$  and  $m'$  denote the conjugates of  $v$  and  $m$  respectively. The action of  $G(M, V)$  on  $\mathbb{H} \times \mathbb{H}$  is free and properly discontinuous. We have a nonsingular surface  $X'(M, V)$  as quotient. This  $X'(M, V)$  is partially compactified by adding a point  $\infty$  into a normal complex space  $X(M, V)$ . Let  $f : Y(M, V) \rightarrow X(M, V)$  be the minimal resolution of  $X(M, V)$ ,  $C$  the exceptional set of  $f$ ,  $\pi : \mathcal{D} \rightarrow Y(M, V)$

the universal covering of  $Y(M, V)$ ,  $C = \pi^{-1}(C)$ . For brevity we denote  $X(M, V)$  and  $Y(M, V)$  by  $X$  and  $Y$  respectively. The space  $X$  has a unique isolated singularity at  $\infty$ , which we call a cusp singularity. The exceptional set  $C$  is a (reduced) cycle of rational curves.

(1.2) Let  $M^*$  be the dual of  $M$ , i.e. by definition  $M^* = \{x \in K; \text{tr}(xy) \in \mathbb{Z} \text{ for any } y \in M\}$ . Define a mapping  $i$  of  $K$  into  $\mathbb{R}^2$  by  $i(x) = (x, x')$ ,  $x \in K$ . Let  $(M^*)^+ = \{x \in M^*; x > 0, x' > 0\}$ , and let  $\Sigma^+(M^*)$  be the convex closure of  $i((M^*)^+)$ ,  $\partial \Sigma^+(M^*)$  be the boundary of  $\Sigma^+(M^*)$ . Then we number lattice points lying on  $\partial \Sigma^+(M^*)$  in a consecutive order. Namely we let  $i^{-1}(\Sigma^+(M^*) \cap i(M^*)) = \{B_j; j \in \mathbb{Z}\}$  with  $B_j < B_k$  for  $j < k$ . The group  $V$  acts on  $M^*$ ,  $\Sigma^+(M^*)$  and  $\partial \Sigma^+(M^*)$ . Let  $v$  be a generator of  $V$  with  $0 < v < 1$ . Then there exists  $s$  such that  $vB_k = B_{k+s}$  for any  $k$ . We know that  $s = -C^2$  by [3]. Moreover there are positive integers  $b_k$  ( $\geq 2$ ) ( $k \in \mathbb{Z}$ ) such that  $b_{k+s} = b_k$  and  $b_k B_k = B_{k-1} + B_{k+1}$  for any  $k \in \mathbb{Z}$ .

§2. Theorem.

Theorem      Let  $T$  be a cusp singularity with  $s \geq 5$ .

Then the space  $\Pi^1$  of infinitesimal deformations of  $T$  is, as a subspace of  $H^1(V, H^0(\mathcal{O}, \theta_{\mathcal{O}}(nC)))$  for  $n$  large enough, generated by

$$\delta_{i,j} := \theta(-iB_j)\delta_j, \quad 0 \leq j \leq s-1, \quad 1 \leq i \leq b_j-1$$

where  $\delta_j = B'_j \partial_1 - B_j \partial_2$ . In particular  $\dim \Pi^1 = s+r$ .

References.

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- [3] Nakamura, I.: Inoue-Hirzebruch Surfaces and a Duality of Hyperbolic Unimodular Singularities, Math. Annalen, 252, 221-235 (1980)